Objective: Find all complex zeros of a polynomial function.

Concept

Fundamental Theorem of Algebra: Every polynomial equation of degree greater than zero has at least one root in the set of complex numbers. Also, the number of complex roots, including those that repeat, is equal to the degree.

$$8x + 5 = 0$$

Degree = 1
Number of complex roots = 1

$$8x^2 + 5 = 0$$

Degree = 2
Number of complex roots = 2

$$x^3 - 8x^2 + 5 = 0$$

Degree = 3
Number of complex roots = 3

$$x^4 - 8x^2 + 5 = 0$$

Degree = 4
Number of complex roots = 4

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Concept

Complex Conjugate Root Theorem

If P is a polynomial equation in one variable with real coefficients, and $a + bi$ is a root of P, with a and b real numbers, then its complex conjugate $a - bi$ is also a root of P.

For example:

If $3 + \sqrt{2}$ is a root, then $3 - \sqrt{2}$ is a root.
If $-1 - 3i$ is a root, then $-1 + 3i$ is a root.
If $-\sqrt{5}$ is a root, then $\sqrt{5}$ is a root.
If $2i$ is a root, then $-2i$ is a root.

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Ex) For each polynomial function: a) find all complex zeros (include multiplicity), b) identify each zero as rational, irrational, or imaginary. Write all zeros in simplest form.

* 4 zeros $f(x) = x^4 - 18x^2 + 32$

① factored form

$f(x) = (x^2 - 16)(x^2 - 2)$

$f(x) = (x+4)(x-4)(x^2-2)$

② zeros

$0 = (x+4)(x-4)(x^2-2)$

$x + 4 = 0$, $\quad x - 4 = 0$, $\quad x^2 - 2 = 0$

$\quad -4 \quad -4 \qquad +4 \quad +4 \qquad +2 \quad +2$

$x = -4 \qquad x = 4 \qquad x^2 = 2$

$\sqrt{x^2} = \pm\sqrt{2}$

$x = -\sqrt{2}, \sqrt{2}$

ⓐ zeros: $-4, 4$; ⓑ rational

$-\sqrt{2}, \sqrt{2}$; irrational

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$g(x) = x^4 - 3x^3 + 5x^2$ ← * 4 zeros

① factored form $g(x) = x^2(x^2 - 3x + 5)$

② zeros $0 = x^2(x^2 - 3x + 5)$

$x \cdot x$

$x^2 = 0$ or $x^2 - 3x + 5 = 0$ ($ax^2 + bx + c = 0$)

$x = 0,\ x = 0$

use Quadratic Formula

$x = \frac{-1b \pm \sqrt{(b)^2 - 4ac}}{2a}$

$a = 1,\ b = -3,\ c = 5$

$x = \frac{-1(-3) \pm \sqrt{(-3)^2 - [4(1)(5)]}}{2(1)}$

$x = \frac{3 \pm \sqrt{9 + {}^-20}}{2}$

$= \frac{3 \pm \sqrt{-11}}{2}$

split

$\sqrt{11} \cdot \sqrt{-1}$ i

$\frac{3}{2} + \frac{\sqrt{-11}}{2},\ \frac{3}{2} - \frac{\sqrt{-11}}{2}$

$\frac{3}{2} + \frac{\sqrt{11}}{2}i,\ \frac{3}{2} - \frac{\sqrt{11}}{2}i$

ⓑ

ⓐ zeros: $0, 0$; rational

$\frac{3}{2} + \frac{\sqrt{11}}{2}i,\ \frac{3}{2} - \frac{\sqrt{11}}{2}i$; imaginary

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* 4 zeros ← $f(x) = x^4 - 16$

$(x^2)^2 - (4)^2$

① factored form

$f(x) = (x^2 + 4)(x^2 - 4)$

② zeros

$0 = (x^2 + 4)(x^2 - 4)$

$x^2 + 4 = 0$ or $x^2 - 4 = 0$

$\quad -4 \quad -4 \qquad\qquad +4 \quad +4$

$x^2 = -4 \qquad\qquad x^2 = 4$

$\sqrt{x^2} = \pm\sqrt{-4} \qquad \sqrt{4}\cdot\sqrt{-1} \qquad \sqrt{x^2} = \pm\sqrt{4}$

$x = -2i, 2i \qquad\qquad x = -2, 2$

ⓐ zeros: $-2, 2$; rational ⓑ

$-2i, 2i$; imaginary

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Ex) For each polynomial function: a) find all complex zeros (include multiplicity), b) identify each zero as rational, irrational, or imaginary. Write all zeros in simplest form.

$$f(x) = 8x^3 + 27$$

$$(2x)^3 + (3)^3$$

① factored form

$$f(x) = (2x + 3)((2x)^2 - (2x)(3) + (3)^2)$$

$$f(x) = (2x + 3)(4x^2 - 6x + 9)$$

② zeros

$$0 = (2x+3)(4x^2 - 6x + 9)$$

$2x + 3 = 0$ or $4x^2 - 6x + 9 = 0$

$$\begin{aligned} -3 \quad &-3 \\ \frac{2x}{2} &= \frac{-3}{2} \\ x &= -\frac{3}{2} \end{aligned}$$

* Use Quadratic Formula

$a = 4$, $b = -6$, $c = 9$

$$x = \frac{-1(-6) \pm \sqrt{(-6)^2 - [4(4)(9)]}}{2(4)}$$

$$x = \frac{6 \pm \sqrt{36 + {}^{-}144}}{8}$$

$$\begin{array}{r} -144 \\ +36 \\ \hline -108 \end{array}$$

$$x = \frac{6 \pm \sqrt{-108}}{8}$$

$$\begin{array}{r} 27 \\ 4\,)\overline{108} \\ -8 \\ \hline 28 \end{array}$$

$$\frac{6}{8} \pm \frac{\sqrt{-108}}{8}$$

$\sqrt{108} \cdot \sqrt{-1}$

$\sqrt{4} \cdot \sqrt{27} \cdot \sqrt{-1}$

$\sqrt{4} \cdot \sqrt{9} \cdot \sqrt{3} \cdot \sqrt{-1}$

$2 \cdot 3 \cdot \sqrt{3} \cdot i$

$6\sqrt{3}$

$$\frac{\cancel{6}^{\,3}}{\cancel{8}_{\,4}} \pm \frac{\cancel{6}^{\,3}\sqrt{3}\,i}{\cancel{8}_{\,4}}$$

$$\frac{3}{4} - \frac{3\sqrt{3}}{4}i \qquad \frac{3}{4} + \frac{3\sqrt{3}}{4}i$$

ⓐ zeros ⓑ

$-\frac{3}{2}$; rational

$\frac{3}{4} - \frac{3\sqrt{3}}{4}i$, $\frac{3}{4} + \frac{3\sqrt{3}}{4}i$ imaginary

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* 3 zeros $\leftarrow$ $f(x) = x^3 + 5x$

① factored form $f(x) = x(x^2+5)$

② zeros $0 = x(x^2+5)$

$x = 0$ or $x^2 + 5 = 0$

$-5 \quad -5$

$x^2 = -5$

$\sqrt{x^{\cancel{2}}} = \pm\sqrt{-5}$

$\sqrt{5}\cdot\sqrt{-1}$

$x = \pm\sqrt{5}\,i$

ⓐ zeros: 0 ; ⓑ rational

$-\sqrt{5}\,i, \sqrt{5}\,i$ imaginary

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Ex) For each polynomial function: a) find all complex zeros (include multiplicity), b) identify each zero as rational, irrational, or imaginary. Write all zeros in simplest form.

$$f(x) = x^4 - 6x^3 - 17x^2 + 102x$$

① factored form

$$f(x) = x(x^3 - 6x^2 - 17x + 102)$$

$$x^2(x-6) - 17(x-6)$$

$$f(x) = x(x-6)(x^2-17)$$

② zeros

$$0 = x(x-6)(x^2-17)$$

$x = 0$, $x - 6 = 0$, $x^2 - 17 = 0$

$+6 \quad +6$ $\qquad$ $+17 \quad +17$

$x = 6$ $\qquad$ $x^2 = 17$

$$\sqrt{x^2} = \pm\sqrt{17}$$

$$x = -\sqrt{17}, \sqrt{17}$$

ⓐ zeros ⓑ

$0, 6$; rational

$-\sqrt{17}, \sqrt{17}$; irrational

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Closure

Irene found the zeros of the polynomial function below. Her work is shown. What is Irene's error? How would you correct the error and what is the correct answer?

$$f(x) = x^4 + 5x^2 - 36$$

$$0 = x^4 + 5x^2 - 36$$
$$(x^2 - 4)(x^2 + 9) = 0$$
$$(x+2)(x-2)(x^2+9) = 0$$
$$x+2=0 \quad x-2=0 \quad x^2+9=0$$
$$x=-2 \quad x=2 \quad x^2=9$$
$$x = \pm\sqrt{9}$$
$$x = \pm 3$$
$$\boxed{\text{zeros: } -2, 2, -3, 3}$$

Irene's error is that she didn't subtract the 9. I would correct the error by writing $x = \pm\sqrt{-9}$ which gives $x = \pm 3i$. The zeros are $-2, 2, -3i$ *and* $3i$.