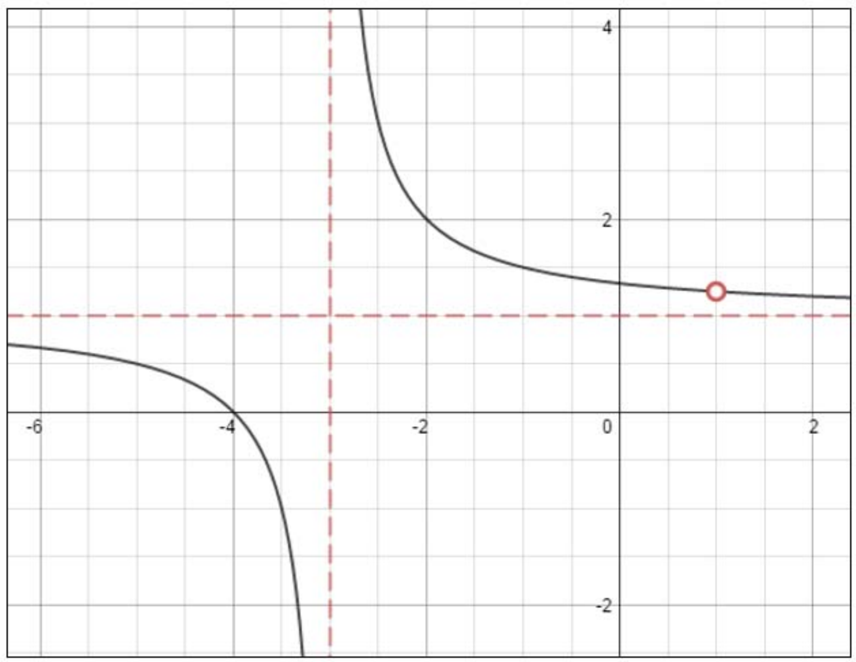
Objective: Find holes, asymptotes, and intercepts of rational functions.

Concept

Holes: If the numerator and denominator of a rational function have a common factor, **a hole** is created in the graph for the value of x that makes the common factor equal to 0. This **is called a point of discontinuity** and is **represented as an open circle** at the corresponding point in the curve.

$$f(x) = \frac{x^2 + 3x - 4}{x^2 + 2x - 3} \rightarrow f(x) = \frac{(x+4)\,(x-1)}{(x+3)\,(x-1)} \rightarrow f(x) = \frac{x+4}{x+3} \; \textit{with a hole at } (1{,}1.25)$$



Objective: Find holes, asymptotes, and intercepts of rational functions.

Concept

Steps to Find Holes of Rational Functions

1. Given a function of the form $f(x) = \frac{p(x)}{q(x)}$, factor both numerator and denominator completely. Reduce common factors.

2. Find the x-coordinate of the hole: There will be a hole at the x-value where the common factor(s) equal 0.

3. Find the y-coordinate of the hole: Plug the hole's x-value into the simplified function.

$$f(x) = \frac{x^2+3x-4}{x^2+2x-3} \rightarrow f(x) = \frac{(x+4)(x-1)}{(x+3)(x-1)} \rightarrow f(x) = \frac{x+4}{x+3}$$

x-coordinate: $x - 1 = 0 \rightarrow x = 1$

y-coordinate: $f(1) = \frac{1+4}{1+3} = \frac{5}{4} = 1.25$

The hole is at $(1,1.25)$

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Concept

Steps to Find Asymptotes of Rational Functions

1. Given a function of the form $f(x) = \frac{p(x)}{q(x)}$, factor both numerator and denominator completely. Reduce common factors.

2. **Find All Vertical Asymptotes**: Using the simplified function, set factors in the denominator equal to 0 and solve for x.

3. **Find the Horizontal Asymptote**: Use the original or simplified function.
   a. Take the ratio of the first terms of the numerator and denominator.
   b. Reduce.
   c. Result is $\frac{c}{x} \rightarrow y = 0$
      Result is $c \rightarrow y = c$

Objective: Find holes, asymptotes, and intercepts of rational functions.

Example)

$$f(x) = \frac{x^2 - 3x - 4}{x^2 - 16}$$

hole (point of discontinuity): $\left(4, \frac{5}{8}\right)$

vertical asymptote(s): $x = -4$

horizontal asymptote: $y = 1$

zero: ____________

y-intercept: ____________

① find the simplified function.

$$f(x) = \frac{(x-4)(x+1)}{(x-4)(x+4)}$$

$$f(x) = \frac{x+1}{x+4}$$

② hole $\left(4, \frac{5}{8}\right)$

ⓐ $x - 4 = 0$

$x = 4$

ⓑ $y = ?$

$f(4) = \frac{4+1}{4+4} = \frac{5}{8}$

③ asymptotes

ⓐ vertical

$x + 4 = 0$

$x = -4$

ⓑ horizontal

$\frac{x^2}{x^2} = 1$ or $\frac{x}{x} = 1$

$y = 1$

Objective: Find holes, asymptotes, and intercepts of rational functions.

Example)

$$f(x) = \frac{x+5}{x^2+5x}$$

hole (point of discontinuity): $\left(-5, -\frac{1}{5}\right)$

vertical asymptote(s): $x = 0$

horizontal asymptote: $y = 0$

zero: ______________

y-intercept: ______________

① find the simplified function

$$f(x) = \frac{x+5}{x(x+5)}$$

$$f(x) = \frac{1}{x}$$

② hole $\left(-5, \frac{-1}{5}\right)$

ⓐ $x = ?$

$x + 5 = 0$

$x = -5$

ⓑ $y = ?$

$f(-5) = \frac{1}{-5}$

$= -\frac{1}{5}$

③ asymptotes

ⓐ vertical

$\star\, x = 0$

ⓑ $\star$ horizontal

$\frac{x}{x^2} = \frac{1}{x}$ or $\frac{1}{x}$

$\star\, y = 0$

Objective: Find holes, asymptotes, and intercepts of rational functions.

Concept

Finding zeros/x-intercepts and the y-intercept of Rational Functions

1. Use the simplified form of the function.

2. **To find zeros/x-intercepts**, let $y = 0$ and solve for x, because the x-intercepts (zeros) are values of x that produce a y-value of 0.

3. **To find the y-intercept**, let $x = 0$ and solve for y, because the y-intercept is the point where the function intersects the y-axis.

Objective: Find holes, asymptotes, and intercepts of rational functions.

Example)

$$f(x) = \frac{x^2 - 3x - 4}{x^2 - 16}$$

hole (point of discontinuity): $\left(4, \frac{5}{8}\right)$

vertical asymptote(s): $x = -4$

horizontal asymptote: $y = 1$

zero: -1

y-intercept: $(0, 1/4)$

$$f(x) = \frac{(x-4)(x+1)}{(x-4)(x+4)} \rightarrow f(x) = \frac{x+1}{x+4}$$

4 zero

$x = ?$ when $y = 0$

$$\underbrace{f(x)}_{0} = \frac{x+1}{x+4}$$

$$x+4 \cdot 0 = \frac{x+1}{\cancel{x+4}} \cdot \cancel{x+4}$$

$0 = x + 1$

$x = -1$ zero

5 y-int.

if $x = 0$, $y = ?$

$$f(0) = \frac{0+1}{0+4} = \frac{1}{4}$$

$(0, 1/4)$ y-int.

Objective: Find holes, asymptotes, and intercepts of rational functions.

Example)

$$f(x) = \frac{x+5}{x^2+5x}$$

hole (point of discontinuity): $\left(-5, -\frac{1}{5}\right)$

vertical asymptote(s): $x = 0$

horizontal asymptote: $y = 0$

zero: none

y-intercept: none

$$f(x) = \frac{x+5}{x(x+5)} \rightarrow f(x) = \frac{1}{x}$$

④ zero

$x = ?$ when $y = 0$

$$f(x) = \frac{1}{x}$$

$$x \cdot 0 = \frac{1}{\cancel{x}} \cdot \cancel{x}$$

$0 = 1 \rightarrow 0 \neq 1$

no zero

⑤ y-int.

if $x = 0$, $y = ?$

$f(0) = \frac{1}{0} \rightarrow$ undefined

no y-int.

Objective: Find holes, asymptotes, and intercepts of rational functions.

Example)

$$\underbrace{f(x) = \frac{2x+1}{x-3}}_{\text{simplified function}}$$

hole (point of discontinuity): none

vertical asymptote(s): $x = 3$

horizontal asymptote: $y = 2$

zero: $-\frac{1}{2}$

y-intercept: $(0, -1/3)$

① hole
no common factor = no hole

② *vertical asym.
$x - 3 = 0$
$\star\ x = 3$

③ *horiz. asy.
$\frac{2x}{x} = 2 \rightarrow {}^{*}y = 2$

④ zero
$x = ?$ when $y = 0$
$f(x) = \frac{2x+1}{x-3}$
$x-3 \cdot 0 = \frac{2x+1}{\cancel{x-3}} \cdot \cancel{x-3}$
$0 = 2x + 1$
$-1 = 2x$
$x = {}^{*}-\frac{1}{2}$ zero

⑤ * y-int.
if $x = 0$, $y = ?$
$f(0) = \frac{2(0)+1}{0-3} = \frac{1}{-3} = \frac{-1}{3}$
$(0, \frac{-1}{3})^{*}$

Objective: Find holes, asymptotes, and intercepts of rational functions.

Example)

$$f(x) = \frac{x-4}{x^2-4x+3}$$

hole (point of discontinuity): none

vertical asymptote(s): $x=1$, $x=3$

horizontal asymptote: $y=0$

zero: 4

y-intercept: $(0, -4/3)$

① simp. function

$$f(x) = \frac{x-4}{(x-3)(x-1)}$$

② hole

no common factor = no hole

③ vert. asy.

$x-3=0$, $x-1=0$

$x=3$ $\quad$ $x=1$

④ * horiz. asy.

$$\frac{x}{x^2} = \frac{1}{x} \rightarrow {}^{*}y=0$$

⑤ * zero

$x=?$ when $y=0$

$$f(x) = \frac{x-4}{(x-3)(x-1)}$$

⑥ * y-int.

if $x=0$, $y=?$

$$f(0) = \frac{0-4}{0^2-4(0)+3} = \frac{-4}{3}$$

$(0, -4/3)$ *

$$(x-3)(x-1)\cdot 0 = \frac{x-4}{\cancel{(x-3)}\cancel{(x-1)}} \cdot \cancel{(x-3)}\cancel{(x-1)}$$

$0 = x-4$

$x=4$ * zero

Objective: Find holes, asymptotes, and intercepts of rational functions.

Example)

$$f(x) = \underbrace{\frac{x^2 - 9}{\sqrt{x+5}}}_{\text{simplified function}}$$

vertical asymptote(s): $x = -5$

y-intercept: $\left(0, \frac{-9\sqrt{5}}{5}\right)$

① vert. asym.

$$\sqrt{x+5} = 0$$

$$\left(\cancel{\sqrt{x+5}}\right)^{\cancel{2}} = 0^2$$

$$x + 5 = 0$$

$$x = -5$$

② y-int.

if $x = 0$, $y = ?$

$$f(0) = \frac{0^2 - 9}{\sqrt{0+5}} = \frac{-9}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{-9\sqrt{5}}{\sqrt{25}} = \frac{-9\sqrt{5}}{5}$$

$$\left(0, \frac{-9\sqrt{5}}{5}\right)$$